Exchange economy with Cobb-Douglas and perfect substitutes

The utility functions of the agents are $u_1(x, y) = x + 2y$, $u_2(x, y) = \sqrt{xy}$ and the initial endowments are $\omega^1 = (1, 2)$, $\omega^2 = (1, 3)$. Find the Walras equilibrium.

Solutions

The demands of the agents are:

$$x^1(p) = \begin{cases} \left(\frac{m_1}{p_1}, 0\right), & \text{if } p_2 > 2p_1; \\ \left(x, \frac{m_1}{p_2} - 2x\right) : 0 \leq x \leq \frac{m_1}{p_2}, & \text{if } p_2 = 2p_1; \\ \left(0, \frac{m_1}{p_2}\right), & \text{if } p_2 < 2p_1. \end{cases}$$

$$x^2(p) = \left(\frac{m_2}{2p_1}, \frac{m_2}{2p_2}\right)$$

Observe that:

$$m_1 = p_1 + 2p_2, \quad m_2 = p_1 + 3p_2$$

Now we need to check three cases: $p_2 > 2p_1$, $p_2 = 2p_1$ and $p_2 < 2p_1$

If $p_2 = 2p_1$, then:

$$m_2 = p_1 + 3p_2 = 7p_2$$

and

$$x^2(p_1, 2p_1) = \left(\frac{7}{2}, \frac{7}{4}\right)$$

and the demand for good 1 exceeds the total resources of that good (2 units). Therefore, $p_2 = 2p_1$ cannot be an equilibrium.

If $p_2 > 2p_1$, then the total demand for good 2 is:

$$x_2^1(p) + x_2^2(p) = 0 + \frac{m_2}{2p_2} = \frac{p_1 + 3p_2}{2p_2} = 4$$
$$p_1 = 5p_2$$

which contradicts that $p_2 > 2p_1$. Therefore, $p_2 > 2p_1$ cannot be an equilibrium either.

Finally, the possibility $p_2 < 2p_1$ remains. In this case, the total demand for good 1 is:

$$x_1^1(p) + x_2^1(p) = 0 + \frac{m_2}{2p_1} = \frac{p_1 + 3p_2}{2p_1} = 2$$

from which we obtain:

$$\frac{p_2}{p_1} = \frac{1}{12}$$